INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 4.12]

EPECTATION ALGEBRA – CENTRAL LIMIT THEOREM

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A. Paper 1 questions (SHORT)

EXPECTATION ALGEBRA

1. [Maximum mark: 15]

Let X and Y be random variables with

$$\mathrm{E}(X)=10,\ \mathrm{E}(Y)=15,\ \mathrm{Var}(X)=4,\ \mathrm{Var}(Y)=9$$

(a) Calculate the following values.

$\mathrm{E}(5X)=$
$\mathrm{E}(X+Y+5)=$
$\mathrm{E}(X-Y+5)=$
$\mathrm{E}(3X+2Y)=$
$\mathrm{E}(3X-2Y)=$

[5]

(b) Calculate the following values.

$\mathrm{Var}(5X)=$
$\mathrm{Var}(X+Y+5)=$
$\mathrm{Var}(X-Y+5)=$
$\mathrm{Var}(3X+2Y)=$
$\mathrm{Var}(3X-2Y)=$

[10]

2. [Maximum mark: 10]

Independent random variables X and Y are such that

$$X \sim \mathrm{Po}(10) \quad \text{and} \quad Y \sim \mathrm{Po}(8)$$

(a) Write down the values of

(i) $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ (ii) $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ [2]

(b) Calculate

(i) $\mathrm{E}(X+Y)$ (ii) $\mathrm{Var}(X+Y)$ [2]

(c) Calculate

(i) $\mathrm{E}(X-Y)$ (ii) $\mathrm{Var}(X-Y)$ [2]

(d) Do $X+Y$ and $X-Y$ follow any of the known distributions? Explain. [2]

(e) **Hence** find the probability $\mathrm{P}(X+Y=20)$ [2]

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3. [Maximum mark: 10]

Independent random variables X and Y are such that

$$X \sim \mathrm{N}(10, 4) \quad \text{and} \quad Y \sim \mathrm{N}(8, 2)$$

(a) Write down the values of

(i) $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ (ii) $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ [2]

(b) Calculate

(i) $\mathrm{E}(X+Y)$ (ii) $\mathrm{Var}(X+Y)$ [2]

(c) Calculate

(i) $\mathrm{E}(X-Y)$ (ii) $\mathrm{Var}(X-Y)$ [2]

(d) Do $X+Y$ and $X-Y$ follow any of the known distributions? Explain. [2]

(e) **Hence** find the probability $\mathrm{P}(X+Y \geq 20)$ [2]

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4. [Maximum mark: 12]

Independent random variables X and Y are such that

$$X \sim \mathrm{B}(10, 0.4) \quad \text{and} \quad Y \sim \mathrm{B}(15, 0.2)$$

(a) Find the values of

(i) $\mathrm{E}(X)$ and $\mathrm{Var}(X)$ (ii) $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$ [4]

(b) Calculate

(i) $\mathrm{E}(X+Y)$ (ii) $\mathrm{Var}(X+Y)$ [2]

(c) Calculate

(i) $\mathrm{E}(X-Y)$ (ii) $\mathrm{Var}(X-Y)$ [2]

(d) Do $X+Y$ and $X-Y$ follow any of the known distributions? [1]

(e) Find the probability $\mathrm{P}(X+Y=1)$ [3]

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5. [Maximum mark: 6]

Independent random variables X and Y are such that $X \sim \mathrm{Po}(10)$ and $Y \sim \mathrm{Po}(8)$.

Let $A = 4X - 5Y$. Calculate $\mathrm{E}(A)$ and $\mathrm{Var}(A)$

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6. [Maximum mark: 6]

Independent random variables X and Y are such that $X \sim \mathrm{N}(10, 4)$ and $Y \sim \mathrm{N}(8, 2)$

Let $A = 4X - 5Y$. Calculate $\mathrm{E}(A)$ and $\mathrm{Var}(A)$

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7. [Maximum mark: 4]

The random variable Y is such that $\mathrm{E}(2Y+3)=6$ and $\mathrm{Var}(2-3Y)=11$.

Calculate (i) $\mathrm{E}(Y)$ (ii) $\mathrm{Var}(Y)$

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8. [Maximum mark: 6]

Independent random variables R and S are such that

$$R \sim \mathrm{N}(5,1) \quad \text{and} \quad S \sim \mathrm{N}(8,2)$$

The random variable V is defined by $V=3S-4R$. Calculate $\mathrm{P}(V>5)$

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9. [Maximum mark: 8]

The n independent random variables $X_1, X_2, ..., X_n$ all have the distribution $\mathrm{N}(\mu, \sigma^2)$.

Find the mean and the variance of

(i) $X_1 + X_2$ (ii) $3X_1$ (iii) $X_1 + X_2 - X_3$ (iv) $\bar{X} = \dfrac{X_1 + X_2 + ... + X_n}{n}$

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10. [Maximum mark: 6]

The independent variables U and V are such that $U \sim \mathrm{N}(66, 5)$ and $V \sim \mathrm{N}(19, 3)$.

Calculate the probability that a randomly selected observation from U is more than three times a randomly selected observation from V.

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11. [Maximum mark: 8]

A shop sells apples, pears and peaches. The weights, in grams, of these three types of fruit may be assumed to be normally distributed with means and standard deviations as given in the following table.

Fruit	**Mean**	**Standard Deviation**
Apples	115	5
Pears	110	4
Peaches	105	3

Allan buys 1 apple and 1 pear while Brian buys 1 peach. Calculate the probability that the combined weight of Alan's apple an pear is greater than twice the weight of Brian's peach.

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CENTRAL LIMIT THEOREM

12. [Maximum mark: 6]

Let X be a random variable with $\mathrm{E}(X) = 35$ and $\mathrm{Var}(X) = 25$.

For a sample of 40 observations of X

(a) Find $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$. [3]

(b) Write a comment for the distribution of $\bar{X}$. [2]

(c) Find the probability that $\mathrm{P}(34 \le \bar{X} \le 36)$. [1]

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13. [Maximum mark: 7]

Let $X \sim \mathrm{Po}(35)$. For a sample of 40 observations of X

(a) Find $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$. [3]

(b) Write a comment for the distribution of $\bar{X}$. [2]

(c) Find the probability that $\mathrm{P}(34 \le \bar{X} \le 36)$. [2]

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14. [Maximum mark: 8]

Let $X \sim \mathrm{B}(50, 0.7)$. For a sample of 40 observations of X .

(a) Find $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$. [5]

(b) Write a comment for the distribution of $\bar{X}$. [2]

(c) Find the probability that $\mathrm{P}(34 \le \bar{X} \le 36)$. [1]

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15. [Maximum mark: 6]

Let $X \sim \mathrm{N}(35, 25)$. For a sample of 40 observations of X

(a) Find $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$. [3]

(b) Write a comment for the distribution of $\bar{X}$. [2]

(c) Find the probability that $\mathrm{P}(34 \le \bar{X} \le 36)$. [1]

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16. [Maximum mark: 8]

Suppose that it is known that a large can of a certain paint will cover, on average, 513.3 square metres of wall with a standard deviation of 31.5 square metres. Find the probability that the mean area of the wall covered by a sample of 40 such cans of paint will be between 510.0 square metres and 520.0 square metres.

B. Paper 2 questions (LONG)

17. [Maximum mark: 11]

Ahmed and Brian live in the same house. Ahmed always walks to school and Brian always cycles to school. The times taken to travel to school may be assumed to be independent and normally distributed. The mean and the standard deviation for these times are shown in the table below.

	Mean (minutes)	Standard Deviation (minutes)
Ahmed Walking	30	3
Brian Cycling	12	2

(a) Find the probability that on a particular day Ahmed takes more than 35 minutes to walk to school. [2]

(b) Brian cycles to school on five successive mornings. Find the probability that the total time taken is less than 70 minutes. [4]

(c) Find the probability that, on a particular day, the time taken by Ahmed to walk to school is more than twice the time taken by Brian to cycle to school. [5]

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18. [Maximum mark: 14]

(a) A hospital specializes in treating overweight patients. These patients have weights that are independently, normally distributed with mean 200 kg and standard deviation 15 kg. The elevator in the hospital will break if the total weight of people inside it exceeds 1150 kg. Six patients enter the elevator. Find the probability that the elevator breaks. [7]

(b) A factory makes life size wax copies of famous people. These famous people have weights that are independently, normally distributed with mean 80 kg and standard deviation 10 kg. The life size copies all have exactly the same weight as the famous person they represent. Twelve copies of one particular famous person are placed in the elevator in the factory. This elevator will also break if the total weight of the copies exceeds 1150 kg. Find the probability that the elevator breaks. [7]

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19. [Maximum mark: 11]

The weights of the oranges produced by a farm may be assumed to be normally distributed with mean 205 grams and standard deviation 10 grams.

(a) Find the probability that a randomly chosen orange weighs more than 200 grams. [2]

(b) Five of these oranges are selected at random to be put into a bag. Find the probability that the combined weight of the five oranges is less than 1 kilogram. [4]

(c) The farm also produces lemons whose weights may be assumed to be normally distributed with mean 75 grams and standard deviation 3 grams. Find the probability that the weight of a randomly chosen orange is more than three times the weight of a randomly chosen lemon. [5]

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20. [Maximum mark: 14]

A shop sells apples and pears. The weights, in grams, of the apple may be assumed to have a $\mathrm{N}(200, 15^2)$ distribution and the weights of the pears, in grams, may be assumed to have a $\mathrm{N}(120, 10^2)$ distribution.

(a) Find the probability that the weight of a randomly chosen apple is more than double the weight of a randomly chosen pear. [8]

(b) A shopper buys 3 apples and 4 pears. Find the probability that the total weight is greater than 1000 grams. [6]

21. [Maximum mark: 15]

The weight of tea in Supermug tea bags has a normal distribution with mean 4.2 g and standard deviation 0.15 g. The weight of tea in Megamug tea bags has a normal distribution with mean 5.6 g and standard deviation 0.17 g.

(a) Find the probability that a randomly chosen Supermug tea bag contains more than 3.9 g of tea. [2]

(b) Find the probability that, of two randomly chosen Megamug tea bags, one contains more than 5.4 g of tea and one contains less than 5.4 g of tea. [4]

(c) Find the probability that five randomly chosen Supermug tea bags contain a total of less than 20.5 g of tea. [4]

(d) Find the probability that the total weight of tea in seven randomly chosen Supermug tea bags is more than the total weight in five randomly chosen Megamug tea bags. [5]

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22. [Maximum mark: 14]

The random variable X has probability distribution Po(8).

(a) (i) Find $\mathrm{P}(X = 6)$.

(ii) Find $\mathrm{P}(X = 6 \mid 5 \le X \le 8)$. [5]

(b) $\bar{X}$ denotes the sample mean of $n > 1$ independent observations from X.

(i) Write down $\mathrm{E}(\bar{X})$ and $\mathrm{Var}(\bar{X})$.

(ii) Hence, give a reason why $\bar{X}$ is not a Poisson distribution. [3]

(c) A random sample of 40 observations is taken from the distribution for X.

(i) Find $\mathrm{P}(7.1 < \bar{X} < 8.5)$.

(ii) Given that $\mathrm{P}(\bar{X} \ge a) = 0.025$, find the value of a.

(iii) Given that $\mathrm{P}(8 - k \le \bar{X} \le 8 + k) = 0.95$ (i.e. limits symmetric about $\mu = 8$), find the value of k. [6]

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